INTERNATIONAL JOURNAL OF BUSINESS AND MANAGEMENT (IJBM)
ISSN: 2815-9330 (Online)

VOL. 5, NO. 1 2026 https://iessociety.org/

Real wages and exploitation: A multi-period analysis of income distribution and productivity

Alberto Benítez Sánchez*

National System of Researchers (Sistema Nacional de Investigadoras e Investigadores), Mexico City, Mexico

*Correspondence: abaxayacatl3@gmail.com

Abstract

This paper investigates the relationship between income distribution, exploitation, and real wages within a multi-year production framework. Specifically, it analyzes how variations in the distribution of national income between wages and profits affect average labor productivity and the real wage. A theoretical model is developed in which productivity growth depends on the profit share, allowing the calculation of real wages and the time required for different income distributions to yield equivalent purchasing power. The analysis shows that, under certain conditions, the same real wage can arise in a situation with exploitation and in one without it. In these cases, as a general rule, the two wage levels involved determine an interval of wage levels where the real wage exceeds that corresponding to the zero-exploitation scenario. The results further demonstrate that the effect of increasing profit, or equivalently the exploitation rate, on real wages depends on the duration of the reference period, productivity growth, and wage share. In some cases, higher profit can coexist with higher real wages due to productivity gains. The study contributes to classical and Marxian distribution theory by extending linear production models to a multi-period setting and by clarifying the dynamic interaction between productivity growth and income distribution. These findings provide new insights into the conditions under which exploitation and improvements in workers' purchasing power may coexist.

Keywords**:** Adam Smith, Exploitation Rate, Karl Marx, Labor Productivity, Real Wage

JEL Classification: B12, B14, B51, B16

DOI: https://doi.org/10.56879/ijbm.v5i1.17

1. Background and Purposes of the Study

According to Smith (1981, 13-275), the annual production of goods depends on a set of interrelated institutions such that the growth of any one of them causes the growth of at least one other institution in the set, and, as a general rule, given enough time, these changes also lead to an increase in the annual production of goods. In Benítez (2024), it is shown, based on Smith's arguments, that within this wealth creation system, average labor productivity is positively related to profit, at least within certain limits. Moreover, different aspects of this relation are studied in the simplest case, when it can be represented by means of a linear function, focusing attention on its relevance for the real wage. In this paper, we continue that investigation by calculating the length of time required for the real wage to be the same for two different distributions of the national income between wages and profit in a given economy, paying particular attention to the case where, at one of these distributions, profit is zero. This case presents the same real wage in two different income distributions, one with exploitation and the other without. It is relevant to a precise appreciation of Marx's concept of exploitation because, as shown originally in the quoted paper, the non-zero exploitation distribution of income limits from below the set of those distributions of income for which the real wage is greater than in the zero-exploitation distribution.

The study of the relation between production and income distribution within the classical and Marxist literature has resulted in numerous publications that consider its different aspects by means of linear production models whose main references are the works of Dmitriev (1974), Leontief (1941), Marx (1990), and Sraffa (1960). As can be seen in the reviews of this literature offered in Kurtz & Salvadori (1995), Isikara & Mokre (2025) and Smith (2025) these models typically encompass a single production cycle carried out using a technique chosen from a fixed set of techniques. This restricts their ability to incorporate the effects of changes in income distribution on the available techniques, since these effects usually only materialize after several production cycles. However, as shown in Benítez (2024), Smith (1981, 13-275) theory of economic growth lays the groundwork for constructing a multi-year production model that allows average labor productivity to be represented as a function of income distribution. We briefly present here, and more formally in the following section, the characteristics of that model required in this paper. We study an economy that carries out a succession of annual production cycles in which only one type of good is obtained, completely consuming certain quantities of labor and the same good in the process. Alternatively, for empirical studies, it is assumed that the net product of the model approximately represents the GDP of a given economy. The reference period consists of a particular sequence of years, and the quantities of the good produced are measured using the average product per worker when profit is equal to zero. Wages and profits are the fractions of net annual product, constant throughout the entire sequence and whose sum equals one, that remunerate labor and capital, respectively. Real wage, on the other hand, is the amount of product that can be purchased with the average wage of a unit of labor in the reference period.

DOI: https://doi.org/10.56879/ijbm.v5i1.17

The relationship between productivity and profit already mentioned in the first paragraph is expressed in the productivity/profit rate, which indicates the growth of average labor productivity per unit of profit, assumed to be constant for a given period and economy. Under these conditions, it is shown that if the rate is less than or equal to one, any increase in profit implies a decrease in the real wage, while if it is greater than one, there is an open interval of possible wage values, the upper limit of which is one and the lower limit of which is less than one, each corresponding to a real wage greater than that at the zero-profit level. Therefore, the real wage reaches its maximum possible level while the corresponding profit and rate of exploitation levels are both greater than zero. Indeed, profit is higher than zero only if the rate of exploitation is also higher, a result originally shown in Okisho (1963) further developed in Morishima (1973, 6) and, with an approach relevant for this text, also in Benítez (2011, 24-27). Moreover, there is a set of possible income distribution scenarios in which profit and real wages grow simultaneously. We should also indicate that, given the reference period and either a productivity/profit rate greater than one or the growth rates of output and employment, the optimal wage level and the lower limit of the aforementioned interval are calculated in each case.

To define the research purposes, it is useful to make two preliminary remarks. First, as already indicated, given a reference period, the unit of measure chosen in Benítez (2024) for the good produced is the amount produced per worker when profit is equal to zero.[1] Therefore, the lower limit mentioned in the preceding section is a wage level less than one for which the real wage is equal to one. Second, when considering a sufficiently long reference period, average labor productivity can grow to any level if its growth rate is greater than zero. Based on these remarks, and returning to the model of the cited paper, in this work we proceed following two lines of enquiry that are complementary to the one we have summarized. On the one hand, given a wage between zero and one and an average labor productivity growth rate greater than zero, we calculate the number of years required for the wage to have a purchasing power equal to one. As a result, we identify the period for which the given wage is the lower limit of the aforementioned wage interval. View from another angle, given the two variables just indicated, we calculate the period required for the real wage to equal the one corresponding to the economy during the same period when profit is equal to zero. The interest of this calculation becomes apparent if we consider that:

I) This period is the one necessary for the real wage corresponding to the given wage to be observed in a situation with exploitation and also in one without it. From the

[1] In that text, it is assumed that when profit is equal to zero the average labor productivity growth rate is also zero. Then, regardless of the duration of the period considered, the real wage is equal to the average labor productivity of the first year. In this paper, we adopt this last quantity as unit of measurement. Both units are equivalent under the assumption just mentioned but the new one is easier to work with when, as will be done in sections 5 and 6 ahead, the assumption is disregarded.

DOI: https://doi.org/10.56879/ijbm.v5i1.17

perspective of the workers' economic interest, which consists of obtaining the highest possible real wage, the two situations are equivalent if we disregard the preference for the present.

II) In the calculated period, due to the higher real wage, the situations determined by the wage levels between the given wage and one, in all of which there is exploitation, are preferable for wage earners to the situation in which there is no exploitation.

III) Due to the constant growth of average labor productivity, the real wage is a monotonically increasing function of the duration of the reference period. Thus, given a starting date for the latter, the calculation we perform allows us to know the date from which any period ending after that date has a real wage higher than that of the zero-profit level.

On the other hand, without substantially affecting the research outcomes summarized here, we show that it is possible to disregard the following assumptions: a) average labor productivity does not increase when profit is zero, b) there is no preference for the present, and c) the productivity/profit rate is constant. After this one, eight more sections follow. Section 2 exposes the model and some properties of the productivity/profit rate. The model, except for the unit of measurement used for the good produced, as well as the rate's properties, with one exception, originally appeared in Benítez (2024) but are slightly modified here for the purposes of this paper. In Section 3, we calculate the average labor productivity corresponding to a given period when the annual growth rate of average labor productivity is greater than zero. In Section 4, given a wage between zero and one and a growth rate of average labor productivity greater than zero, we calculate the number of years required for the real wage to equal that corresponding to the same period when profit is zero. This is done by adopting assumptions a), b, and c) just mentioned. The following four sections revisit the same problem, disregarding assumption a) in Section 5 and assumptions a) and b) in Section 6. Section 7 illustrates some results from sections 2 and 6 while Section 8 discusses the productivity/profit rate as a function of the wage. The final section presents some concluding remarks.

2. The Model

This section presents the essential aspects of the reference model originally introduced in Benítez (2024) and some of its main results, adapting them to the unit of measurement for the quantities of the good produced used here, which differs from that of the cited text.

2.1 Basic Concepts

We consider an economy that carries out a succession of annual production cycles in which only one type of good is obtained, completely consuming certain quantities of labor and the same good in the process. The amounts used and produced of the good, as well as the volume of work carried

DOI: https://doi.org/10.56879/ijbm.v5i1.17

out, may vary in all years. For the purpose of studying the effect of changes in the distribution of income on the amounts indicated, we shall assume that throughout the reference period, at the end of each year, wage-earners receive as labor compensation the same fraction w of the corresponding net product. Since this variable determines the fraction of net product constituting profit $(1-w)$, it also affects the volume of investment and employment, particularly investment in science and technology, and thus average labor productivity. On the basis of these assumptions, we now define the relevant variables of our analysis as functions of w.

To each year corresponds a particular index t such that $t = 1,2, \ldots$ For each t, $L_t(w)$ is the labor used, $Q_t(w)$ the quantity produced, $K_t(w)$ the quantity consumed and $Y_t(w)$ the net product $(Q_t(w) - K_t(w))$ obtained producing the good in the year t. Therefore, the average labor productivity in that year and in the period t_1-t_2, which runs from the beginning of the year t_1 to the end of the year t_2 are, respectively:

$$ALP_t(w) = \frac{Y_t(w)}{L_t(w)} \tag{1}$$

$$ALP_{t_1-t_2}(w) = \frac{\sum_{t=t_1}^{t_2} Y_t(w)}{\sum_{t=t_1}^{t_2} L_t\,(w)}. \tag{2}$$

We assume that the two variables just defined are greater than zero for every $w \in]0,1]$. The good is measured using as a unit of measurement the average net product per labor unit in the first year of the reference period and we assume that, in a given period and economy, this quantity is the same for all the levels of w. Therefore, we have:

$$ALP_{t_1}(w) = 1\ \forall\ w \in]0,1]. \tag{3}$$

This assumption facilitates the exposition without affecting the relevant results and, due to the fact that the data corresponding to $ALP_{t_1}(w)$ is normally available in the statistical information only for a single level of $w < 1$, it also facilitates illustrating the results by means of empirical examples.

To calculate the real wage, we notice that the total quantity of the good that can be purchased by the wage-earners in a given year t or in the period t_1-t_2 results from multiplying by w the numerator in the right side of equations (1) and (2), respectively. To obtain the real wage in each case we divide the product by the corresponding quantity of labor. From this conclusion, it follows that in a given year t:

$$s_t(w) = wALP_t(w) \tag{4}$$

and, in the period from t_1 to t_2:

$$s_{t_1-t_2}(w) = wALP_{t_1-t_2}(w) \tag{5}$$

DOI: https://doi.org/10.56879/ijbm.v5i1.17

⇒

$$ALP_{t_1-t_2}(w) = \frac{s_{t_1-t_2}(w)}{w} \; \forall\, w \in]0,1]. \tag{6}$$

2.2 The Productivity/Profit Rate

To study the relation between average labor productivity and profit, we consider the simplest case, when it can be represented by means of a linear function. For this reason, to each volume of profit corresponds an increase in the average labor productivity (relative to its level when $w = 1$) which is in constant proportion $\mu_{t_1-t_2} \geq 0$ to profit. Thus, given a particular economy and a reference period $t_1 - t_2$:

$$\mu_{t_1-t_2} = \frac{ALP_{t_1-t_2}(w) - ALP_{t_1-t_2}(1)}{1-w} \; \forall\, w \in]0,1[. \tag{7}$$

As can be seen in this equation, the newly defined quotient, which we refer to as the productivity/profit rate, is expressed in units of the good per unit of labor and indicates the growth of the average labor productivity per unit of profit.

Equation (7) implies that:

$$ALP_{t_1-t_2}(w) = ALP_{t_1-t_2}(1) + \mu_{t_1-t_2}(1-w). \tag{8}$$

From equations (5) and (8) it follows that:

$$s_{t_1-t_2}(w) = wALP_{t_1-t_2}(1) + w\mu_{t_1-t_2}(1-w). \tag{9}$$

For the purpose of studying the real wage as a function of the distribution of income, we derive this equation with respect to w, which results in:

$$s_{t_1-t_2}{}'(w) = ALP_{t_1-t_2}(1) + \mu_{t_1-t_2}(1-w) - w\mu_{t_1-t_2} \tag{10}$$

$$= ALP_{t_1-t_2}(1) + \mu_{t_1-t_2} - 2\mu_{t_1-t_2}w \tag{11}$$

$$= ALP_{t_1-t_2}(1) + \mu_{t_1-t_2}(1-2w). \tag{12}$$

It is important to distinguish between the two cases characterized respectively by inequalities (13) and (14) discussed next:

$$\mu_{t_1-t_2} \leq ALP_{t_1-t_2}(1). \tag{13}$$

In this case, we can observe in equation (12) that $s_{t_1-t_2}{}'(w) > 0$ for every $w \in [0,1]$ except if $\mu_{t_1-t_2} = ALP_{t_1-t_2}(1)$ and $w = 1$ in which case $s_{t_1-t_2}{}'(w) = 0$. This means that any increase in profit causes a decrease in real wage. Consequently, the real wage peaks when $w = 1$.

$$\mu_{t_1-t_2} > ALP_{t_1-t_2}(1). \tag{14}$$

DOI: https://doi.org/10.56879/ijbm.v5i1.17

In this case, it is possible to observe in equation (12), on the one hand, that $s_{t_1-t_2}'(w)$ is a decreasing monotonic function of w. On the other hand, that $s_{t_1-t_2}'(0) > 0$ while $s_{t_1-t_2}'(1) < 0$. It follows from these remarks that there is only one value of $w \in]0,1[$ for which $s_{t_1-t_2}'(w) = 0$ and also that with this value of w the function $s_{t_1-t_2}(w)$ reaches a maximum. Therefore, $s_{t_1-t_2}(w)$ increases when w decreases from $s_{t_1-t_2}(1) = ALP_{t_1-t_2}(1)$ until it reaches its maximum value for a $w** \in]0,1[$, that we will call the optimal wage share, from which it decreases until $s_{t_1-t_2}(0) = 0$. When it decreases towards zero, it adopts a value $w* \in]0, w**[$ such that:

$$s_{t_1-t_2}(w*) = ALP_{t_1-t_2}(1). \tag{15}$$

See Figure 2 in Section 5. The following proposition demonstrates three formulas relating some variables introduced in this section.

Proposition 1. Let $w* \in]0,1[$ and $t_1 - t_2$ be a period for which equation (15) holds. Then, it is also true that:

$$\mu_{t_1-t_2} = ALP_{t_1-t_2}(w*) \tag{16}$$

$$w** = \frac{ALP_{t_1-t_2}(1) + \mu_{t_1-t_2}}{2\mu_{t_1-t_2}} \tag{17}$$

$$w* = \frac{ALP_{t_1-t_2}(1)}{\mu_{t_1-t_2}}. \tag{18}$$

Proof. By substituting the first and second terms in the numerator of equation (7) with their respective equivalents according to equations (6) and (15), we obtain:

$$\mu_{t_1-t_2} = \frac{\frac{s_{t_1-t_2}(w*)}{w*} - s_{t_1-t_2}(w*)}{1-w*} \tag{19}$$

$$= \frac{s_{t_1-t_2}(w*)\left(\frac{1}{w*} - 1\right)}{1-w*} \tag{20}$$

$$= \frac{s_{t_1-t_2}(w*)\left(\frac{1-w*}{w*}\right)}{1-w*} \tag{21}$$

$$= \frac{s_{t_1-t_2}(w*)}{w*}. \tag{22}$$

Equations (6) corresponding to $w*$ and (22), taken together, imply equation (16).

To calculate $w**$, we set the right side of equation (12) equal to zero:

$$0 = ALP_{t_1-t_2}(1) + \mu_{t_1-t_2}(1 - 2w**) \tag{23}$$

DOI: https://doi.org/10.56879/ijbm.v5i1.17

$\Rightarrow$

$$2w ** \mu_{t_1-t_2} = ALP_{t_1-t_2}(1) + \mu_{t_1-t_2}. \tag{24}$$

This result implies equation (17).

Finally, equation (5) corresponding to $w *$ and equation (15), taken together, imply that:

$$w * ALP_{t_1-t_2}(w *) = ALP_{t_1-t_2}(1) \tag{25}$$

$\Rightarrow$

$$w *= \frac{ALP_{t_1-t_2}(1)}{ALP_{t_1-t_2}(w *)}. \tag{26}$$

Substituting $ALP_{t_1-t_2}(w *)$ with the left side of equation (16) we obtain equation (18).

3. Average Labor Productivity Over a Period

In this section, we calculate the average labor productivity over a period $t_1 - t_2$ by knowing the average annual growth rate of labor productivity (g) over that period. The average product per worker in the first year is $ALP_{t_1}(w)$ units, in the second year it is $ALP_{t_1}(w)(1+g)$ units, in the third year $ALP_{t_1}(w)\ (1+g)^2$ units and so on. The average labor productivity over the period is the quotient of the sum of the quantities produced per worker divided by the number of years of the period:

$$ALP_{t_1-t_2}(w) = \frac{ALP_{t_1}(w)[1 + (1+g) + \ (1+g)^2 \ldots + (1+g)^{t_2-t_1}]}{t_2 - t_1 + 1}. \tag{27}$$

It is possible to observe that the second factor in the numerator of the right side in this equation is a sum of a geometric progression containing $t_2 - t_1 + 1$ terms whose first term is 1 and whose common ratio is $(1+g)$. By making:

$$n = t_2 - t_1 + 1 \tag{28}$$

$$G = 1 + g \tag{29}$$

we can substitute $ALP_{t_1}(w)$, $t_2 - t_1 + 1$ and $1 + g$ in equation (27) according to equations (3), (28) and (29) respectively. Then, we obtain:

$$ALP_{t_1-t_2}(w) = \frac{1 + G + \ G^2 \ldots + G^{t_2-t_1}}{n}. \tag{30}$$

Applying the formula for the sum of a geometric progression, yields:

$$ALP_{t_1-t_2}(w) = \frac{\left(\dfrac{G^n - 1}{G - 1}\right)}{n} \tag{31}$$

DOI: https://doi.org/10.56879/ijbm.v5i1.17

where the average labor productivity during the reference period appears as a multiple of the average labor productivity in the first year of the period. For what follows, it is important to add that equations (5) and (31) allow the real wage of the reference period to be expressed in the following way:

$$s_{t_1-t_2}(w) = w\frac{\left(\frac{G^n-1}{G-1}\right)}{n}. \tag{32}$$

Example 1. According to Weinstock (2023) in the United States between 1949 and 2021 the average annual growth rate of labor productivity (g) in the non-farm business sector was 2.1%, and, according to York (2023), the average share of wages in national income from 1929 to 2023 was 69.9%. We will calculate the real wage for the period beginning in 1949, which lasts 33.520329 years (33 years plus 365 × 0.520329 days).

Equation (28) allows us to calculate that the period considered ends in 1981.520329. Substituting each variable in formula (32) with the corresponding value yields:

$$s_{1949-1981.520329}(0.699) = (0.699)\frac{\left(\frac{(1.021)^{33.520329}-1}{1.021-1}\right)}{33.520329} \tag{33}$$

$$= 0.999944 \tag{34}$$

This shows that the real wage for the period under consideration is equal to the average labor productivity in the first year of that period, except for a small difference due to rounding in the calculations. It is also equal to the real wage for the same period when profit is zero, assuming that, in this case, the growth rate of labor productivity is zero. The choice of the period is justified later, in Example 2.

Table 1. Tabulation of $s_{1949-t_2}(w)$ for two different pairs (w, g)

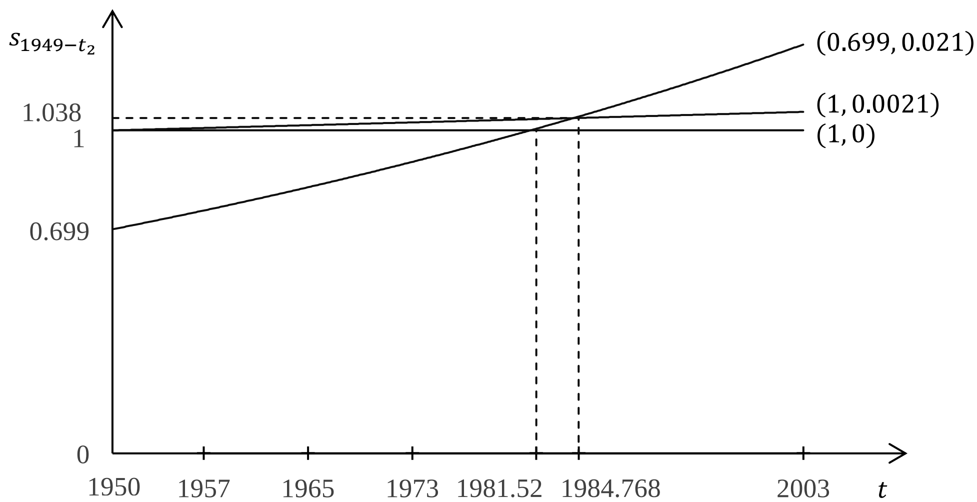
t_2	$(1, 0.0021)$	$(0.699, 0.021)$	t_2	$(1, 0.0021)$	$(0.699, 0.021)$
1950	1	0.699	1981.52	1.03411	1
1957	1.0073	0.7529	1984.768	1.038	1.038
1965	1.0159	0.82065	1993	1.046	1.1312
1973	1.0245	0.8969	2003	1.0577	1.277

Source: own elaboration with data from Weinstock (2023) and York (2023).

DOI: https://doi.org/10.56879/ijbm.v5i1.17

Table 1 presents the average labor productivity values for two different wage levels and their corresponding labor productivity growth rates. These values were calculated for eight reference periods of varying lengths, all starting in 1949. Figure 1 presents the graphs of the functions tabulated in Table 1. There, we can appreciate the evolution of average labor productivity for three different pairs (w, g) as a function of the number of years included in the reference period. In particular, it shows that if $(w, g) = (0.699, 0.021)$ the average real wage in any period starting in 1949 and ending either after 1981.52 or after 1984.768 is greater to the one corresponding to the system in the same period if $(w, g) = (1, 0)$ or $(w, g) = (1, 0.0021)$, respectively.

Figure 1. Graphics of $s_{1949-t_2}(w)$ for three different pairs (w, g)



4. Years Required for the Real Wage to Equal a Given Amount of Product

In this Section, given a wage $w \in]0,1[$ and a growth rate of average labor productivity $g > 0$, we calculate the length of the period necessary for the real wage to be equal to one. To this end, we solve a more general problem, when it is required for the real wage to equal any given quantity Q greater than the given wage.

It follows from equation (32) that the number of years the reference period must last for the real wage of that period to equal Q satisfies the following equation:

$$Q = w \frac{\left(\frac{G^n - 1}{G - 1}\right)}{n}. \tag{35}$$

DOI: https://doi.org/10.56879/ijbm.v5i1.17

To solve this equation for n, we will use a property of Lambert W function. With this purpose, we first perform some transformations of equation (35) to bring it to the appropriate form. The last equation implies that:

$$\frac{nQ}{w} = \frac{G^n - 1}{G - 1} \tag{36}$$

⇒

$$\frac{nQ(G-1)}{w} = G^n - 1 \tag{37}$$

⇒

$$G^n = \frac{nQ(G-1)}{w} + 1 \tag{38}$$

⇒

$$1 = \left(\frac{nQ(G-1)}{w} + 1\right) G^{-n}. \tag{39}$$

Multiplying both sides by $\frac{w}{Q(G-1)}$ we get:

$$\frac{w}{Q(G-1)} = \left(n + \frac{w}{Q(G-1)}\right) G^{-n}. \tag{40}$$

Multiplying both sides by $-G^{-\frac{w}{Q(G-1)}}$ yields:

$$\frac{w}{Q(G-1)}\left(-G^{-\frac{w}{Q(G-1)}}\right) = \left(n + \frac{w}{Q(G-1)}\right)\left(-G^{-\frac{w}{Q(G-1)}}\right) G^{-n} \tag{41}$$

⇒

$$-\frac{w}{Q(G-1)}\left(G^{-\frac{w}{Q(G-1)}}\right) = \left(-n - \frac{w}{Q(G-1)}\right) G^{-n-\frac{w}{Q(G-1)}}. \tag{42}$$

Replacing the second factor on the right side with its equivalent $e^{(lnG)\left(-n-\frac{w}{Q(G-1)}\right)}$ we obtain:

$$-\frac{w}{Q(G-1)}\left(G^{-\frac{w}{Q(G-1)}}\right) = \left(-n - \frac{w}{Q(G-1)}\right) e^{(lnG)\left(-n-\frac{w}{Q(G-1)}\right)}. \tag{43}$$

Multiplying both sides by lnG results in:

$$-\frac{w}{Q(G-1)}\left(G^{-\frac{w}{Q(G-1)}}\right)(lnG) = (lnG)\left(-n - \frac{w}{Q(G-1)}\right) e^{(lnG)\left(-n-\frac{w}{Q(G-1)}\right)}. \tag{44}$$

Applying the Lambert W function to both sides of this equation, we can write:

DOI: https://doi.org/10.56879/ijbm.v5i1.17

$$W\left[-\frac{w}{Q(G-1)}\left(G^{-\frac{w}{Q(G-1)}}\right)(lnG)\right]=W\left[(lnG)\left(-n-\frac{w}{Q(G-1)}\right)e^{(lnG)\left(-n-\frac{w}{Q(G-1)}\right)}\right]. \quad (45)$$

Using the distinctive property of Lambert W function, we can substitute the right side of this equation with its equivalent, obtaining:

$$W\left[-\frac{w}{Q(G-1)}\left(G^{-\frac{w}{Q(G-1)}}\right)(lnG)\right]=(lnG)\left(-n-\frac{w}{Q(G-1)}\right) \quad (46)$$

$\Rightarrow$

$$-n-\frac{w}{Q(G-1)}=\frac{W\left[-\frac{w}{Q(G-1)}\left(G^{-\frac{w}{Q(G-1)}}\right)(lnG)\right]}{lnG} \quad (47)$$

$\Rightarrow$

$$n=-\frac{w}{Q(G-1)}-\frac{W\left[-\frac{w}{Q(G-1)}\left(G^{-\frac{w}{Q(G-1)}}\right)(lnG)\right]}{lnG}. \quad (48)$$

This brings us to the solution of the proposed problem.

Example 2. Using the data from Example 1, we will calculate the number of years a reference period beginning in 1949 must encompass for the real wage to equal the average productivity per worker in that year.

Substituting each variable in formula (48) with the corresponding value, we obtain:

$$n=-\frac{0.699}{(1)(0.021)}-\frac{W\left[\left(-\frac{0.699}{(1)(0.021)}\right)\left(1.021^{-\frac{0.699}{(1)(0.021)}}\right)(0.020783)\right]}{0.020783} \quad (49)$$

$$=-33.285714-\frac{W(-0.346367)}{0.020783} \quad (50)$$

Since $-0.346367\in\left]-\frac{1}{e},0\right[$ there are two values for $W(-0.346367)$:[2] -0.691805 and -1.38843. Introducing the first of these into equation (50) yields:

$$n=-33.285714-\frac{-0.691798}{0.020783} \quad (51)$$

$$=0.001347 \quad (52)$$

[2] The calculator used is at the address https://www.had2know.org/academics/lambert-w-function-calculator.html

DOI: https://doi.org/10.56879/ijbm.v5i1.17

This is not the correct answer, which can be seen from the fact that it is a period of less than one year. Indeed, during the first year, the real wage is less than one.

Now, introducing the second value of $W(-0.346367)$ into equation (50) results in:

$$n = -33.285714 - \frac{-1.38843}{0.020783} \tag{53}$$

$$= 33.520329 \tag{54}$$

This is the correct answer, as proved in Example 1.

5. Years Required for the Real Wage of Two Different Wage Levels to be Equal

In this Section, given two pairs of wage levels with their corresponding growth rate of average labor productivity (w_0, g_0) and (w_1, g_1) such that $0 < w_0 < w_1 \leq 1$ and $g_0 > g_1 > 0$, we calculate the length of the period necessary for the real wage to be same with the two pairs.

Let $G_0 = 1 + g_0$ and $G_1 = 1 + g_1$. It follows from equation (32) that the number of years the reference period must last for the real wage of that period to be the same at the two indicated wage levels satisfies the following equation:

$$w_0 \frac{\left(\frac{{G_0}^n - 1}{G_0 - 1}\right)}{n} = w_1 \frac{\left(\frac{{G_1}^n - 1}{G_1 - 1}\right)}{n}. \tag{55}$$

Multiplying both sides of this equation by n results in:

$$\frac{w_0({G_0}^n - 1)}{G_0 - 1} = \frac{w_1({G_1}^n - 1)}{G_1 - 1} \tag{56}$$

$\Rightarrow$

$$w_0(G_1 - 1){G_0}^n - w_0(G_1 - 1) = w_1(G_0 - 1){G_1}^n - w_1(G_0 - 1) \tag{57}$$

$\Rightarrow$

$$w_0(G_1 - 1){G_0}^n - w_1(G_0 - 1){G_1}^n = w_0(G_1 - 1) - w_1(G_0 - 1). \tag{58}$$

It is important to note that this formula cannot be used when $g_1 = 0$, since in this case a division by zero would be involved, as can be seen on the right side of equation (55). Although we don't solve equation (58) for n in general here, we show with an example how n can be obtained based on the problem data, which is sufficient for the purposes of this paper.

Example 3. Considering the data from Example 1, we assume that, when profit is equal to zero, average labor productivity grows at a rate that is one-tenth of the rate indicated in that example. Thus, we will calculate n when $w_0 = 0.699$, $w_1 = 1$, $g_0 = 0.021$ and $g_1 = 0.0021$.

Substituting the variables in equation (58) with the corresponding values, we obtain:

DOI: https://doi.org/10.56879/ijbm.v5i1.17

$$(0.699)(0.0021)(1.021)^n - (1)(0.021)(1.0021)^n$$
$$= (0.699)(0.0021) - (1)(0.021) \tag{59}$$

Entering this equation into the Wolfram equation solver results in $n = 36.768$.[3] Given this result, equation (28) allows us to calculate that the period considered ends in 1984.768. Substituting successively the variables on the left and right sides of equation (55) with the corresponding values gives, respectively:

$$(0.699)\frac{\left(\frac{(1.021)^{36.768}-1}{1.021-1}\right)}{36.768} = 1.038486 \tag{60}$$

$$(1)\frac{\left(\frac{(1.0021)^{36.768}-1}{1.0021-1}\right)}{36.768} = 1.038486 \tag{61}$$

The results of the last two calculations allow us to verify the duration of the period obtained. Furthermore, they indicate the real wage when $w = 0.699$ and $w = 1$, respectively. Both measured using the productivity per worker in the first year of the reference period.

Furthermore, the average labor productivity with the wage $w = 0.699$, according to formula (6), is:

$$ALP_{1949-1984.768}(0.699) = \frac{1.038486}{0.699} \tag{62}$$

$$= 1.485673 \tag{63}$$

Therefore, according to Proposition 1:

$$\mu_{1949-1984.768} = 1.485673 \tag{64}$$

To calculate the optimal wage level, we substitute the variables with their corresponding values in the equation (17):

$$w^{**} = \frac{1.038486 + 1.485673}{(1.485673)(2)} \tag{65}$$

$$= 0.84950 \tag{66}$$

To calculate the average labor productivity and the real wage when the wage attains its optimal level, we substitute the variables with their corresponding values in equations (8) and (5), respectively:

$$ALP_{1949-1984.768}(0.84950) = 1.038486 + 1.485673(1 - 0.84950). \tag{67}$$

[3] The equation solver is accessible with the link https://www.wolframalpha.com/calculators/equation-solver-calculator

DOI: https://doi.org/10.56879/ijbm.v5i1.17

$$= 1.262079 \quad (68)$$

$$s_{1949-1984.768}(0.84950) = (0.84950)(1.262079) \quad (69)$$

$$= 1.0721367 \quad (70)$$

Table 2. Tabulation of two functions of the wage when $\mu_{1949-1984.768} = 1.485673$

w	$ALP_{1949-1984.768}(w)$	$s_{1949-1984.768}(w)$	w	$ALP_{1949-1984.768}(w)$	$s_{1949-1984.768}(w)$
0	2.48	0	0.775	1.37	1.06
0.17	2.27	0.38	0.849	1.26	1.072
0.35	2.003	0.7	0.925	1.149	1.06
0.52	1.75	0.91	1	1.038	1.038
0.699	1.48	1.038			

Source: own elaboration with data from Weinstock (2023) and York (2023).

Figure 2. Two functions of the wage when $\mu_{1949-1984.768} = 1.485673$

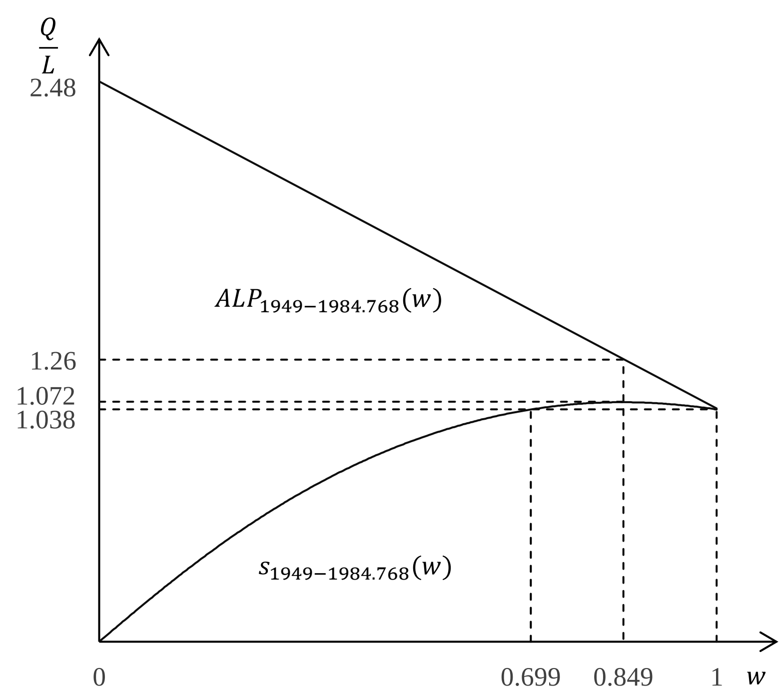


Table 2 presents the values of the average labor productivity and the real wage for nine different wage levels. These values were calculated for a reference period whose productivity/profit rate is

DOI: https://doi.org/10.56879/ijbm.v5i1.17

1.485673. Figure 2 presents the graphs of the functions tabulated in Table 2. In it we can appreciate the interval $]0.699, 1[$ containing the wage levels for which the real wage is greater than the one corresponding to the situation where there is no exploitation. Also, that the set of possible wage levels is divided in two semi open intervals $[0, 0.849[$ and $]0.849, 1]$ integrated by those that are smaller and greater than the optimal wage level, respectively. Considering the set of wage earners as a class, they find their economic interest either when the wage increases within the first interval or when it decreases within the second one. At the same time, the exploitation rate decreases and increases in the first and in the second case, respectively.

6. The Present Value of the Real Wage over a Period

In this section, we will revisit the problems discussed in sections 3 and 5, this time considering the present value at the beginning of the reference period of the quantities involved in each case.

6.1 The Present Value of the Average Labor Product of a Sequence of Years

Let $h \geq 0$ be the annual discount rate, which we assume to be constant throughout the reference period. The present value of the average product per worker in the first year is $\frac{ALP_{t_1}(w)}{(1+h)}$ units, in the second year it is $\frac{ALP_{t_1}(w)(1+g)}{(1+h)^2}$ units, in the third year $\frac{ALP_{t_1}(w)(1+g)^2}{(1+h)^3}$ units, and so on. The present value of the average labor productivity over the period is the quotient of the sum of the present values of the quantities produced each year per worker divided by the number of years in the period:

$$PVALP_{t_1-t_2}(w) = \frac{ALP_{t_1}(w)\left[\frac{1}{1+h} + \frac{1+g}{(1+h)^2} + \frac{(1+g)^2}{(1+h)^3} \dots + \frac{(1+g)^{t_2-t_1}}{(1+h)^{t_2-t_1+1}}\right]}{t_2 - t_1 + 1} \tag{71}$$

$$= \frac{\frac{ALP_{t_1}(w)}{1+h}\left[1 + \frac{1+g}{1+h} + \frac{(1+g)^2}{(1+h)^2} \dots + \frac{(1+g)^{t_2-t_1}}{(1+h)^{t_2-t_1}}\right]}{t_2 - t_1 + 1}. \tag{72}$$

Let

$$H = \frac{1+g}{1+h}. \tag{73}$$

Substituting in equation (72) variables $ALP_{t_1}(w)$, $t_2 - t_1 + 1$ and $\frac{1+g}{1+h}$ according to equations (3), (28) and (73), respectively, we get:

$$PVALP_{t_1-t_2}(w) = \frac{\frac{1}{1+h}(1 + H + H^2 \dots + H^{t_2-t_1})}{n}. \tag{74}$$

DOI: https://doi.org/10.56879/ijbm.v5i1.17

Applying the formula for a sum of a geometric progression, we can write the last equation as follows:

$$PVALP_{t_1-t_2}(w) = \left(\frac{1}{1+h}\right)\frac{\left(\frac{H^n-1}{H-1}\right)}{n} \tag{75}$$

where the average labor productivity during the reference period appears as a multiple of the average labor productivity in the first year of the period.

Equations (5) and (75) allow us to express the present value of the real wage of the reference period by means of the next formula:

$$PVs_{t_1-t_2}(w) = \left(\frac{w}{1+h}\right)\frac{\left(\frac{H^n-1}{H-1}\right)}{n}. \tag{76}$$

It follows from equations (73), (74) and (76) that the present values of average labor productivity and average real wage over a period are both increasing, constant or decreasing functions of the length of the period if g is greater, equal or less than h, respectively.

6.2 Years Required for the Present Value of the Real Wage to be the Same for Two Different Wage Levels

Let w_0, w_1 $(0 < w_0 < w_1 \leq 1)$ be a given pair of wage levels, g_0 and g_1 $(g_0 > g_1 > 0)$ are the growth rates of average labor productivity when $w = w_0$ and $w = w_1$, respectively. Furthermore, let $H_0 = \frac{1+g_0}{1+h}$ and $H_1 = \frac{1+g_1}{1+h}$.

It follows from equation (76) that the number of years the reference period must last for the present value of the real wage of that period to be the same at the two indicated wage levels satisfies the following equation:

$$\left(\frac{w_0}{1+h}\right)\frac{\left(\frac{{H_0}^n-1}{H_0-1}\right)}{n} = \left(\frac{w_1}{1+h}\right)\frac{\left(\frac{{H_1}^n-1}{H_1-1}\right)}{n}. \tag{77}$$

Multiplying both sides of this equation by $n(1+h)$ results in:

$$\frac{w_0({H_0}^n-1)}{H_0-1} = \frac{w_1({H_1}^n-1)}{H_1-1} \tag{78}$$

$\Rightarrow$

$$w_0(H_1-1){H_0}^n - w_0(H_1-1) = w_1(H_0-1){H_1}^n - w_1(H_0-1) \tag{79}$$

$\Rightarrow$

$$w_0(H_1-1){H_0}^n - w_1(H_0-1){H_1}^n = w_0(H_1-1) - w_1(H_0-1). \tag{80}$$

DOI: https://doi.org/10.56879/ijbm.v5i1.17

It is important to note that this formula cannot be used when $g_0 = h$ nor when $g_1 = h$ because in either of these cases a division by zero would be involved, as can be seen in equation (77). Although we don't solve equation (80) for n in general here, we show through an example how n can be obtained based on the problem data, which is sufficient for the purposes of this paper.

7. Study of a Particular Case Illustrating Sections 2 and 6

In this Section, we study a particular case with the purpose of illustrating some results from sections 2 and 6. Revisiting the case from examples 1 and 3, we will also take into account that according to YCHARTS (2025), the long-term average discount rate in the USA is 2.3%.

7.1 The Value of n in Equation (80)

We will calculate n when $w_0 = 0.699$, $w_1 = 1$, $g_1 = 0.021, g_0 = 0.0021$ and $h = 0.023$. Substituting each variable with its value in equation (80), we obtain:

$$(0.699)\left(\frac{1.0021}{1.023}-1\right)\left(\frac{1.021}{1.023}\right)^n-(1)\left(\frac{1.021}{1.023}-1\right)\left(\frac{1.0021}{1.023}\right)^n$$
$$=(0.699)\left(\frac{1.0021}{1.023}-1\right)-(1)\left(\frac{1.021}{1.023}-1\right). \quad (81)$$

Performing some of the operations indicated in this equation yields:

$$(0.699)(-0.0204301075)(0.99804496)^n-(-0.001955034)(0.97956998)^n$$
$$=(0.699)(-0.0204301075)-(-0.001955034). \quad (82)$$

Entering this equation into the Wolfram equation solver results in $n = 42.7478$. Given this result, equation (28) allows us to calculate that the period considered ends in 1990.7478. Substituting successively the variables on the left and right sides of equation (77) with the corresponding data gives, respectively:

$$\left(\frac{0.699}{1.023}\right)\frac{\left(\frac{\left(\frac{1.021}{1.023}\right)^{42.7478}-1}{\frac{1.021}{1.023}-1}\right)}{42.7478}=\left(\frac{0.699}{1.023}\right)\frac{\left(\frac{(0.99804496)^{42.7478}-1}{-0.001955034}\right)}{42.7478} \quad (83)$$

$$=0.656128390 \quad (84)$$

$$\left(\frac{1}{1.023}\right)\frac{\left(\frac{(0.97956998)^{42.7478}-1}{-0.0204301075}\right)}{42.7478}=0.6561285 \quad (85)$$

The results of the last two calculations allow us to verify the duration of the period obtained. Furthermore, they indicate the present values the of real wage when $w = 0.699$ and $w = 1$, respectively. Both measured using the product per worker in the first year of the reference period.

DOI: https://doi.org/10.56879/ijbm.v5i1.17

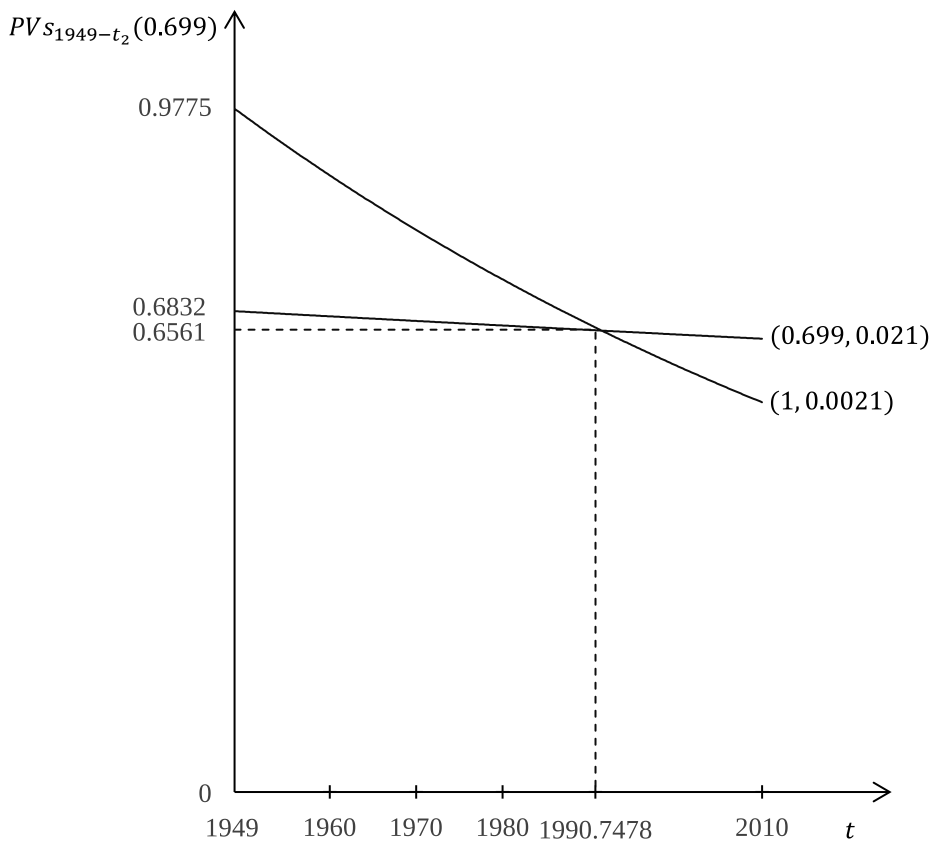
Table 3 shows the present value of average labor productivity for two different wage levels and their corresponding labor productivity growth rates. These values were calculated for seven reference periods of varying lengths, all starting in 1949.

Table 3. Tabulation of $PV\ s_{1949-t_2}(w)$ for two different pairs (w, g)

t_2	$(1, 0.0021)$	$(0.699, 0.021)$	t_2	$(1, 0.0021)$	$(0.699, 0.021)$
1949	0.9775	0.6832	1990.7478	0.6561	0.6561
1960	0.8748	0.6759	2000	0.6065	0.6503
1970	0.7938	0.6694	2010	0.5571	0.6440
1980	0.7228	0.6571			

Source: own elaboration with data from Weinstock (2023), YCHARTS (2025) and York (2023)

Figure 3. Graphics of $PVs_{1949-t_2}(0.699)$ for two pairs (w, g)



DOI: https://doi.org/10.56879/ijbm.v5i1.17

Figure 3 presents the graphs of the functions tabulated in Table 3. There, we can appreciate the evolution of the present value of average labor productivity as a function of the number of years included in the reference period. The figure illustrates the last case mentioned in the paragraph just below equation (76), when $g < h$, for two different pairs (w, g).

7.2 The Productivity/Profit Rate in the Period 1949-1990.7478

First, we calculate $ALP_{1949-1990.7478}(1)$ and $ALP_{1949-1990.7478}(0.699)$ to then obtain the productivity/profit rate for the period. Successively substituting the variables in equation (31) with their corresponding values yields:

$$ALP_{1949-1990.7478}(1) = \frac{\left(\frac{1.0021^{42.7478} - 1}{1.0021 - 1}\right)}{42.7478} \tag{86}$$

$$= 1.04511 \tag{87}$$

$$ALP_{1949-1990.7478}(0.699) = \frac{\left(\frac{1.021^{42.7478} - 1}{1.021 - 1}\right)}{42.7478} \tag{88}$$

$$= 1.5943 \tag{89}$$

Substituting the variables with their respective values in formula (7) yields:

$$\mu_{1949-1990.7478} = \frac{1.5943 - 1.04511}{1 - 0.699} \tag{90}$$

$$= 1.8247 \tag{91}$$

These results allow us to calculate $w*$ and $w**$ by substituting the variables with their corresponding values in equations (18) and (17), respectively:

$$w* = \frac{1.04511}{1.8247} \tag{92}$$

$$= 0.5727 \tag{93}$$

$$w** = \frac{1.04511 + 1.82472}{(1.82472)(2)} \tag{94}$$

$$= 0.78637 \tag{95}$$

7.3 The Present Value of $s_{t_1-t_2}(w)$

To study the present value of the real wage $PVs_{t_1-t_2}(w)$, it is useful to consider that g is a monotonically decreasing function of w, and therefore the inverse function $w = w(g)$ exists. Expressing some functions of w as functions of g simplifies the construction of Table 4, which allows us to obtain the graph of the function $PVs_{t_1-t_2}(w)$.

DOI: https://doi.org/10.56879/ijbm.v5i1.17

Table 4. Tabulation of four functions of g when $\mu_{1949-1990.7478} = 1.8247$

g	$ALP_{1949-1990.7478}(g)$	$w(g)$	$s_{1949-1990.7478}(g)$	$PVs_{1949-1990.7478}(g)$
0.0021	1.0454	1	1.0454	0.6561
0.0081	1.1892	0.9211	1.095	0.6737
0.0141	1.3595	0.8278	1.1253	0.6781
–	1.4351	0.7863	1.1285	–
0.021	1.5943	0.699	1.1144	0.6571
0.025	1.7531	0.6121	1.073	0.6234
0.03	1.979	0.4883	0.9663	0.5522
0.035	2.2402	0.3451	0.7731	0.4347
0.04	2.5424	0.1795	0.4563	0.2526
0.044	2.8182	0.028	0.0789	0.0431

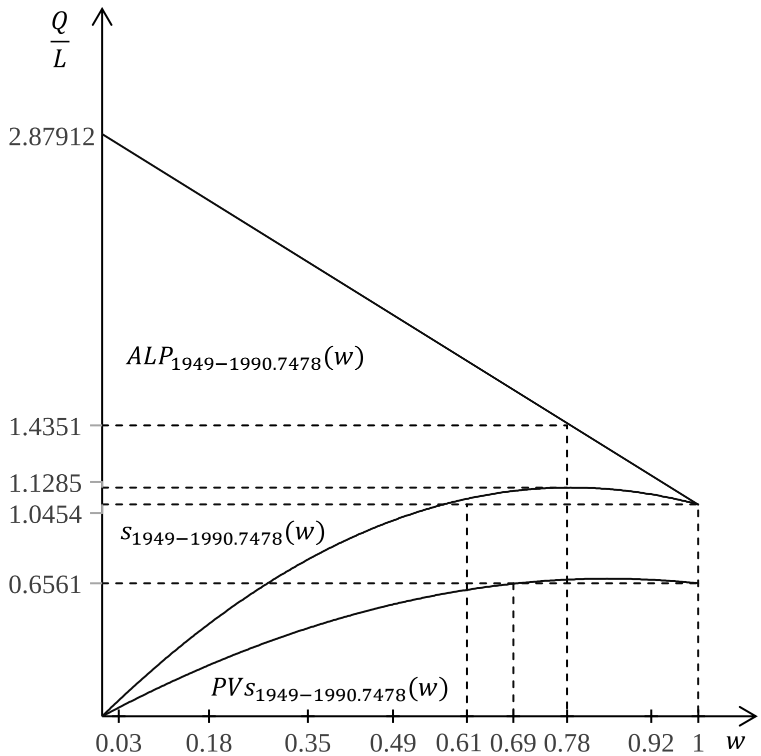
Source: own elaboration with data from Weinstock (2023), YCHARTS (2025) and York (2023).

The table shows the values of several variables corresponding to ten different levels of g for a period of 42.7478 years when the productivity/profit rate is 1.8247. To construct Table 4, we begin by selecting the value of g corresponding to $w = 1$ and, through trial and error, a value of g for which $ALP_{1949-1990.7478}(g)$ is close to 2.8247, that is, to the value of this function when $w = 0$. We then choose the other values within the interval determined by the two already selected. Next, we successively enter the values from the first column of that table into equation (31), thus obtaining the corresponding values from the second column. By then entering successively these values, along with two other known values, into equation (8) and solving it for w, we obtain the corresponding values from the third column. Finally, by entering the values from each row of columns one and three into equations (32) and (76), we obtain the corresponding values from the fourth and fifth columns, respectively. Figure 4 presents the graphs of $s_{1949-1990.7478}(w)$, $PVs_{1949-1990.7478}(w)$ and $ALP_{1949-1990.7478}(w)$, built with the data shown in Table 4. In the first graph, we can appreciate the interval $]0.61, 1[$ containing the wage levels for which the real wage is greater than the one corresponding to the situation where there is no exploitation. Also, that the set of possible wage levels is divided in two semi open intervals $[0, 0.78[$ and $]0.78, 1]$ integrated

DOI: https://doi.org/10.56879/ijbm.v5i1.17

by those that are smaller and greater than the optimal wage level, respectively. Considering the set of wage earners as a class, they find their economic interest either when the wage increases within the first interval or when it decreases within the second one. At the same time, the exploitation rate decreases and increases in the first and in the second case, respectively. Similar conclusions may be reached observing the second graph. Although the quantities involved are different in each case.

Figure 4. Three variables as functions of *w*



8. The Productivity/Profit Rate as a Function of w

In the general case, the productivity/profit rate is not necessarily the same for all values of w and can even take on negative values if average labor productivity decreases sufficiently when profit increases. Indeed, it follows from equation (7) that this occurs if, for a level of w less than one, the average labor productivity for the period is lower than when w equals one.

According to the above, we can express $\mu_{t_1-t_2}$ as a function of the wage as follows:

$$\mu_{t_1-t_2} = \mu_{t_1-t_2}(w) \; \forall \; w \in [0,1[. \tag{96}$$

To obtain the change in the real wage that takes place when the wage falls from one to a level $w \in [0,1[$ we subtract from $ALP_{t_1-t_2}(1)$ the right side of equation (9), after rewriting it according to the equation (96):

DOI: https://doi.org/10.56879/ijbm.v5i1.17

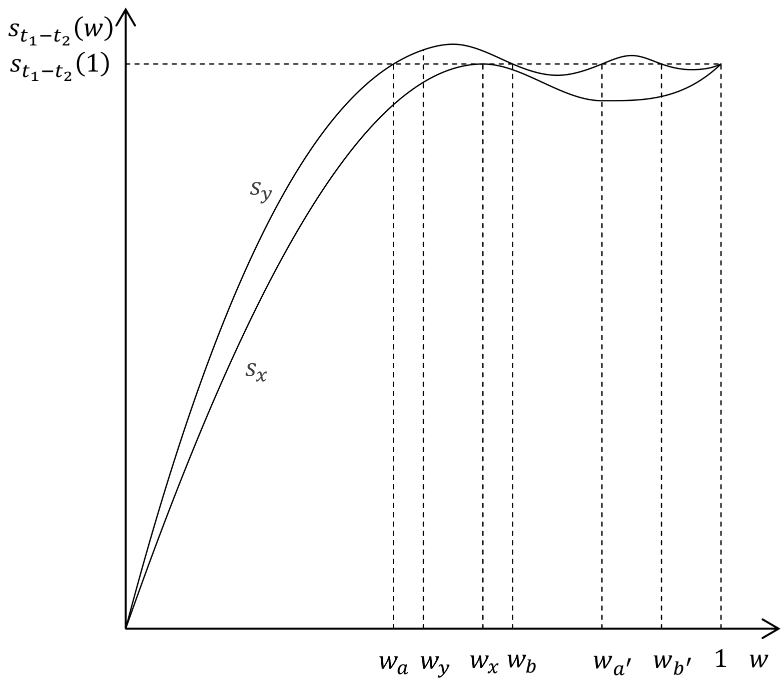
$$\Delta s_{t_1-t_2}(w) = ALP_{t_1-t_2}(1) - wALP_{t_1-t_2}(1) - w\mu_{t_1-t_2}(w)(1-w) \tag{97}$$

$$= ALP_{t_1-t_2}(1)(1-w) - w\mu_{t_1-t_2}(w)(1-w) \tag{98}$$

$$= \left[ALP_{t_1-t_2}(1) - w\mu_{t_1-t_2}(w)\right](1-w). \tag{99}$$

Considering the sign on the right side of this equation, we must distinguish the two cases indicated below in inequalities (100) and (101)

Figure 5. Two possible forms of $s_{t_1-t_2}(w)$



$$ALP_{t_1-t_2}(1) \geq w\mu_{t_1-t_2}(w)\ \forall\ w \in [0,1[. \tag{100}$$

In this case, $\Delta s_{t_1-t_2}(w) \geq 0$, so $ALP_{t_1-t_2}(1) \geq s_{t_1-t_2}(w)\ \forall\ w \in [0,1[$. The equality holds for those values w_x in the indicated interval where the graph of the function $s_{t_1-t_2}(w)$ is tangent to the horizontal line of height equal to one, as illustrated by the curve s_x in Figure 5. For all other wage levels within the interval, the real wage is less than $s_{t_1-t_2}(1)$. This is the rare case where a $w \in [0,1[$ whose real wage equals $s_{t_1-t_2}(1)$ does not limit from below an open interval where the real wage is greater than $s_{t_1-t_2}(1)$ for all wage levels in the interval. Therefore, the maximum level of the real wage is $s_{t_1-t_2}(1)$, but it can be reached by more than one wage level.

$$ALP_{t_1-t_2}(1) < w\mu_{t_1-t_2}(w) \tag{101}$$

for at least one wage level $w_y \in]0,1[$.

DOI: https://doi.org/10.56879/ijbm.v5i1.17

In this case, $\Delta s_{t_1-t_2}(w_y) < 0$, so $ALP_{t_1-t_2}(1) < s_{t_1-t_2}(w_y)$. That is, the real wage is higher at w_y than when all income goes to workers. Since $s_{t_1-t_2}(w)$ is a continuous function, w_y is contained in an open interval $]w_a, w_b[$ such that to each wage level within it corresponds a real wage greater than $s_{t_1-t_2}(1)$. Furthermore, there may be more than one interval with this characteristic, as illustrated by the curve s_Y in Figure 5. It follows from the above that the real wage reaches its highest level at one or more wage levels smaller than one.

9. Conclusions

As can be seen in the sections 5, 6 and 7, by disregarding successively only assumption a) and then assumptions a) and b) indicated in Section 1 and proceeding with the corresponding calculation, longer periods are required for the same real wage to be observed in both exploitative and non-exploitative situations. This in itself does not substantially alter any of the three properties of these situations described in the same section. In fact, the length of these periods can vary considerably depending on the different variables involved. However, as shown in Section 8, when the productivity/profit rate is a function of the wage, the first and the last of those properties are not modified, but the second one is. Indeed, if this rate is not constant, then, given a period and economy such that the same real wage corresponds to a pair $(w, 1)$ where $w < 1$, there may be several intervals of wage levels for which the corresponding real wages are either above or below the real wage at the zero-profit level. In this case, there would also be more than one wage level lower than one for which the corresponding real wage is equal to that at the zero-profit level.

References

Benítez Sánchez, A. (2011). Distribution, exploitation and profit rates. *Cuadernos de economía*, vol. 30, no. 54, pp. 1-19.

https://revistas.unal.edu.co/index.php/ceconomia/article/view/24101/24731

Benítez Sánchez, A. (2024). The Optimal Wage Share in Adam Smith. *Theoretical Economics Letters*, vol. 14, 978-999. https://doi.org/10.4236/tel.2024.143051

Dmitriev, V. K. (1974). *Economic essays on value, competition and utility*. United Kingdom, Cambridge University Press. English translation of Dmitriev essays published in 1904 in Russian, edited by D. M. Nuti.

Isikara, G. & Mokre, P. (2025). *Marx's Theory of Value at the Frontiers: Classical Political Economics, Imperialism and Ecological Breakdown*. New York, Routledge.

Kurz, H., & Salvadori, N. (1995). *Theory of production*. United Kingdom, Cambridge University Press. https://www.researchgate.net/publication/227389612_Theory_of_Production

DOI: https://doi.org/10.56879/ijbm.v5i1.17

Leontief, W. (1941). *The structure of the American economy 1919-1929*. Cambridge, Harvard University Press. https://babel.hathitrust.org/cgi/pt?id=mdp.39015023177895&seq=199

Marx, K. (1990). *Capital Volume I*. London, Penguin Books. English translation by Ernest Mandel of the German-Language book *Das Kapital* edited by the Institute for Marxism-Leninism of the Central Committee of the SED.

https://www.surplusvalue.org.au/Marxism/Capital%20%20Vol.%201%20Penguin.pdf

Morishima, M. (1973). *Marx's economics*. Cambridge, Cambridge University Press.

http://digamo.free.fr/morishimarx.pdf

Okisho, N. (1963). A mathematical note on Marxian theorems, *WeltwirtshaftlishesArchiv*, no. 91, pp. 287-299. https://pdfcoffee.com/a-mathematical-reformulation-of-marxian-economic-theorems-regarding-value-exchange-value-surplus-value-and-the-transformation-problem-pdf-free.html

Smith, A. (1981). *An inquiry into the nature and causes of the wealth of nations*, Volume I. Indianapolis, Liberty Found, Inc. Reprint of Vol. II of *The Glasgow edition of the works and correspondence of Adam Smith* published by Oxford University Press in 1976.

Smith, M. (2025). Technological Progress in the Theory of Accumulation of the Classical Economist and Marx. *History of economic ideas*, vol. XXXIII, no. 1, pp. 9-35.

Sraffa, P. (1960). *Production of commodities by means of commodities. Prelude to a critique of economic theory*. United Kingdom, Cambridge University Press.

http://pombo.free.fr/sraffa60.pdf

Weinstock, L. R. (2023). Introduction to U. S. Economy: Productivity. *In Focus, Congressional Research Service.* https://crsreports.congress.gov/product/pdf/IF/IF10557

YCHARTS (2025). Discount rate. https://ycharts.com/indicators/us_discount_rate

York, E. (2023). Labor Share of Net Income is Within its Historical Range. *Tax Foundation.*

https://taxfoundation.org/blog/labor-share-net-income-within-historical-range/

DOI: https://doi.org/10.56879/ijbm.v5i1.17